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Bounds for A−G, A−H, G−H, and a Family of Inequalities of Ky Fan's Type, Using a General Method

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Upper and lower bounds are found for $A-G$ and $G-H$ and lower bounds for $A-H$. The method of proof we use appears to be widely applicable and so to illustrate this we also prove a generalization of the classical Ky Fan inequality. © 2000 Academic Press

1. INTRODUCTION

Let A_n, G_n, and H_n be the arithmetic, geometric, and harmonic means of the positive numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ formed with the positive weights w_k $(k = 1, 2, \ldots, n)$ whose sum is unity. Our first objective is to prove the following inequalities:

$$\frac{1}{2x_1}\sum_1^n w_k(x_k - \mathrm{A}_n)^2 \geq \mathrm{A}_n - \mathrm{G}_n \geq \frac{1}{2x_n}\sum_1^n w_k(x_k - \mathrm{A}_n)^2 \tag{1.1}$$

$$\frac{1}{2x_1}\sum_1^n w_k(x_k - \mathrm{G}_n)^2 \geq \mathrm{A}_n - \mathrm{G}_n \geq \frac{1}{2x_n}\sum_1^n w_k(x_k - \mathrm{G}_n)^2 \tag{1.2}$$

$$\mathrm{A}_n - \mathrm{H}_n \geq \frac{1}{2x_n}\sum_1^n w_k(x_k - \mathrm{A}_n)^2 \tag{1.3}$$

$$\mathrm{A}_n - \mathrm{H}_n \geq \frac{1}{2x_n}\sum_1^n w_k(x_k - \mathrm{H}_n)^2 \tag{1.4}$$

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$$\frac{G_n}{2x_1^2}\sum_1^n w_k(x_k - \mathrm{G}_n)^2 \geq \mathrm{G}_n - \mathrm{H}_n \geq \frac{\mathrm{G}_n}{2x_n^2}\sum_1^n w_k(x_k - \mathrm{G}_n)^2 \quad (1.5)$$

$$\frac{G_n}{2x_1^2}\sum_1^n w_k(x_k - \mathrm{H}_n)^2 \geq \mathrm{G}_n - \mathrm{H}_n \geq \frac{\mathrm{G}_n}{2x_n^2}\sum_1^n w_k(x_k - \mathrm{H}_n)^2. \quad (1.6)$$

Any one of these inequalities is strict unless all the x_k are equal.

The result (1.1) was proved by D. I. Cartwright and M. J. Field in 1978 [1] and recently H. Alzer [2] proved the right hand side of (1.2) which improved the right-hand side of (1.1). We think that the other inequalities are new. Of course, they are not all independent; already we have noted that (R1.2) strengthens (R1.1). (We use R and L to denote right and left.) The "tightest" of these inequalities are (R1.2), (R1.4), (R1.6), (L1.1), and (L1.5). We have been unable to find left hand sides of these types, for (1.3) and (1.4).

The methods of proof used in [1, 2] were similar and involved a novel combination of mathematical induction and the Lagrange multiplier method. The technique presented here is quite different from that of those authors and it is capable of proving all of (1.1)–(1.6). The method also appears to have wide application and we shall illustrate this feature by proving the following result:

THEOREM 1. *With* $p \geq 1$ *let* $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 2^{-(1/p)}$ *and let* y_k *be positive numbers defined by* $x_k^p + y_k^p = 1$.
Then

$$\mathrm{A}_n(x)\mathrm{G}_n(y) \geq \mathrm{A}_n(y)\mathrm{G}_n(x). \quad (1.7)$$

This inequality is strict unless all the x_k *are equal.*

In case $p = 1$, (1.7) is the classical Ky Fan inequality. There are many proofs of this result and we refer the reader to [3, 4, 5, 6], for example. Several more references are also to be found in these sources. When $p > 1$ we believe that (1.7) is new.

It would not be difficult to enlarge on the number of inequalities which can be proved by the technique described here but we shall content ourselves with the above.

2. THE PROOFS OF (1.1)–(1.6)

Since the technique is the same for all the inequalities in (1.1)–(1.6) we shall prove one of them in full and then indicate how the other proofs would proceed.

Although (R1.1) is one of the weaker inequalities we shall illustrate the method by proving it. All the features of the method are present in this proof and it is the most concise.

Proof of (R1.1). Let us suppose, initially, that $x_1 < x_2 < \cdots < x_n$ and write

$$Q_n \equiv \mathrm{A}_n - \mathrm{G}_n - \frac{1}{2x_n}\sum_1^n w_k(x_k - \mathrm{A}_n)^2, \tag{2.1}$$

where, of course,

$$\mathrm{A}_n = \sum_1^n w_k x_k \qquad \text{and} \qquad \mathrm{G}_n = \prod_1^n x_k^{w_k}.$$

In Q_n we keep $x_1, x_2, \ldots, x_{n-1}$ fixed and allow x_n to vary in the interval $(x_{n-1}, +\infty)$. We differentiate Q_n with respect to x_n getting

$$\begin{aligned}\frac{dQ_n}{dx_n} = w_n - \frac{w_n}{x_n}\mathrm{G}_n - \frac{1}{x_n}\sum_{k\neq n} w_k(x_k - \mathrm{A}_n)(0 - w_n)\\ - \frac{1}{x_n}w_n(x_n - \mathrm{A}_n)(1 - w_n) + \frac{1}{2x_n^2}\sum_1^n w_k(x_k - \mathrm{A}_n)^2.\end{aligned}$$

Because $\sum_1^n w_k(x_k - \mathrm{A}_n) = 0$ this reduces to

$$\frac{dQ_n}{dx_n} = \frac{w_n}{x_n}[\mathrm{A}_n - \mathrm{G}_n] + \frac{1}{2x_n^2}\sum_1^n w_k(x_k - \mathrm{A}_n)^2 \tag{2.2}$$

and, since this is strictly positive, Q_n increases strictly with x_n in $(x_{n-1}, +\infty)$. So if we write the expression in (2.1) as

$$Q_n = \mathrm{A}_n - \mathrm{G}_n - \frac{1}{2x_n}S_n,$$

and let x_n tend to x_{n-1} (from the right), we find that

$$Q_n > Q_{n-1}. \tag{2.3}$$

In this A_{n-1}, G_{n-1}, and S_{n-1} mean

$$\sum_{k=1}^{n-2} w_k x_k + (w_{n-1} + w_n) x_{n-1}$$

$$\left[\prod_{k=1}^{n-2} x_k^{w_k}\right] x_{n-1}^{(w_{n-1}+w_n)}$$

and

$$\sum_{k=1}^{n-2} w_k (x_k - A_{n-1})^2 + (w_{n-1} + w_n)(x_{n-1} - A_{n-1})^2,$$

respectively.

Now Q_{n-1} involves the positive numbers $x_1 < x_2 < \cdots < x_{n-1}$ with weights $w_1, w_2, \ldots, w_{n-2}, (w_{n-1} + w_n)$ in a way exactly analogous to Q_n and we can proceed as before, keeping all of the x_k fixed except x_{n-1}. We find that (2.3) holds again with n replaced by $n - 1$. Continuing in this way we get

$$Q_n > Q_{n-1} > \cdots > Q_1.$$

Here the meanings of A_{n-p}, G_{n-p}, and S_{n-p} are

$$A_{n-p} = \sum_{k=1}^{n-p-1} w_k x_k + \left[\sum_{k=n-p}^{n} w_k\right] x_{n-p}$$

$$G_{n-p} = \left[\prod_{k=1}^{n-p-1} x_k^{w_k}\right] x_{n-p}^{[\sum_{k=n-p}^{n} w_k]} \tag{2.4}$$

and

$$S_{n-p} = \sum_{k=1}^{n-p-1} w_k (x_k - A_{n-p})^2 + \left[\sum_{k=n-p}^{n} w_k\right] (x_{n-p} - A_{n-p})^2,$$

where $p = 0, 1, 2, \ldots, n-1$ (and we take sums to be zero and products to be unity when the lower limit exceeds the upper limit).

But from (2.4) we see that $Q_1 = 0$ and so, for the case $x_1 < x_2 < \cdots < x_n$, the proof of the right hand side inequality in (1.1) (holding with strict inequality) is complete.

Note. In order to relax $x_1 < x_2 < \cdots < x_n$ to $x_1 \leq x_2 \leq \cdots \leq x_n$ we can proceed as before but with the following modification which we illustrate for the case of the first step (i.e., proceeding from Q_n to Q_{n-1}).

If $x_{n-1} < x_n$ the analysis is the same as before and we get $Q_n > Q_{n-1}$. But if $x_{n-1} = x_n$ then, instead of taking the limit, we simply replace x_n by x_{n-1} and obtain $Q_n = Q_{n-1}$. In any case $Q_n \geq Q_{n-1}$.

Proceeding in this way we get

$$Q_n \geq Q_{n-1} \geq \cdots \geq Q_1$$

and hence $Q_n \geq 0$ which is the right hand side of (1.1).

The argument in the previous paragraph also shows that, if even one of the inequalities in the chain $x_1 \leq x_2 \leq \cdots \leq x_n$ is strict, then we have $Q_n > 0$. This proves that the inequality on the right hand side of (1.1) is strict unless all the x_k are equal. This concludes the proof of the right hand side of (1.1).

The above argument followed lines which could be described as "compression from the right" and this method will deal with all the right hand sides of (1.1)–(1.6). We merely mention that for the inequalities (1.2)–(1.6) the analogues of (2.2) are

$$\frac{dQ_n}{dx_n} = \frac{w_n}{x_n^2} \mathrm{G}_n[\mathrm{A}_n - \mathrm{G}_n] + \frac{1}{2x_n^2} \sum_{k=1}^{n} w_k [x_k - \mathrm{G}_n]^2 > 0 \tag{2.5}$$

$$\frac{dQ_n}{dx_n} = w_n \left[\frac{\mathrm{A}_n}{x_n} - \left[\frac{\mathrm{H}_n}{x_n} \right]^2 \right] + \frac{1}{2x_n^2} \sum_{k=1}^{n} w_k [x_k - \mathrm{A}_n]^2 > 0 \tag{2.6}$$

(because $H_n/x_n < 1$ and $A_n > H_n$)

$$\begin{aligned} \frac{dQ_n}{dx_n} &= w_n \left[\frac{\mathrm{H}_n}{x_n} - \left[\frac{\mathrm{H}_n}{x_n} \right]^2 \right] + \frac{w_n \mathrm{H}_n^2}{x_n^3} [\mathrm{A}_n - \mathrm{H}_n] \\ &\quad + \frac{1}{2x_n^2} \sum_{k=1}^{n} w_k [x_k - \mathrm{H}_n]^2 > 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned} \frac{dQ_n}{dx_n} &= \frac{w_n}{x_n^2} [\mathrm{G}_n^2 - \mathrm{H}_n^2] + \frac{w_n \mathrm{G}_n^2}{x_n^3} [\mathrm{A}_n - \mathrm{G}_n] \\ &\quad + \frac{\mathrm{G}_n}{x_n^3} \left[1 - \frac{w_n}{2} \right] \sum_{k=1}^{n} w_k [x_k - \mathrm{G}_n]^2 > 0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{dQ_n}{dx_n} &= \frac{w_n}{x_n^2} [\mathrm{G}_n - \mathrm{H}_n] + \frac{w_n \mathrm{G}_n \mathrm{H}_n^2}{x_n^4} [\mathrm{A}_n - \mathrm{H}_n] \\ &\quad + \frac{\mathrm{G}_n}{x_n^3} \left[1 - \frac{w_n}{2} \right] \sum_{k=1}^{n} w_k [x_k - \mathrm{H}_n]^2 > 0, \end{aligned} \tag{2.9}$$

respectively.

In all these cases the relaxation of the assumption $x_1 < x_2 < \cdots < x_n$ proceeds exactly as in the note above. And the remark concerning strict inequality except when all the x_k are equal is also valid in these cases.

Turning to the left hand sides of the inequalities (1.1)–(1.6) we follow a similar argument, which could be described as "compression from the left." Let us consider the left hand side of (1.5), for example. We write

$$V_n = \frac{\mathrm{G}_n}{2x_1^2}\sum_1^n w_k(x_k - \mathrm{G}_n)^2 - \mathrm{G}_n + \mathrm{H}_n \equiv \frac{\mathrm{G}_n}{2x_1^2}S_n - \mathrm{G}_n + \mathrm{H}_n \text{ (say)}$$

and assume initially that $x_1 < x_2 < \cdots < x_n$. We keep all the x_k fixed except for $x_1 \in (0, x_2)$. Differentiating with respect to x_1 and simplifying we get

$$\begin{aligned}\frac{dV_n}{dx_1} &= \frac{w_1}{x_1^2}\left[\mathrm{H}_n^2 - \mathrm{G}_n^2\right] + \frac{w_1\mathrm{G}_n^2}{x_1^3}[\mathrm{G}_n - \mathrm{A}_n]\\ &\quad - \frac{\mathrm{G}_n}{x_1^3}\left[1 - \frac{w_1}{2}\right]\sum_{k=1}^n w_k[x_k - \mathrm{G}_n]^2 < 0 \qquad (2.10)\end{aligned}$$

and so V_n decreases strictly as x_1 increases. Hence we let $x_1 \to x_2$ (from the left) and get

$$V_n > V_{n-1},$$

where we now use the notations

$$\begin{aligned}\mathrm{G}_{n-p} &= x_{p+1}^{[\sum_{k=1}^{p+1} w_k]}\prod_{k=p+2}^n x_k^{w_k},\\ \mathrm{H}_{n-p} &= \left[\left[\sum_{k=1}^{p+1} w_k\right]x_{p+1}^{-1} + \sum_{k=p+2}^n w_k x_k^{-1}\right]^{-1}, \qquad (2.11)\end{aligned}$$

and

$$S_{n-p} = \left[\sum_{k=1}^{p+1} w_k\right]\left(x_{p+1} - \mathrm{G}_{n-p}\right)^2 + \sum_{k=p+2}^n w_k\left(x_k - \mathrm{G}_{n-p}\right)^2.$$

Note. The notations in (2.11) are different from those in (2.4), although it is convenient to use the same letters.

Continuing in the same way, as before, we get

$$V_n > V_{n-1} > \cdots > V_1 = 0$$

which completes the proof in case $x_1 < x_2 < \cdots < x_n$.

The relaxation of this condition proceeds as before and the conclusion that the left hand inequality in (1.5) is strict unless all the x_k are equal is arrived at in the same way.

The other left hand sides of these inequalities are proved similarly. For brevity we leave the details in each case to the reader.

Note. The expression in (2.10) differs from that in (2.8) by having the opposite sign and by having x_1 and w_1 instead of x_n and w_n. This difference is generally the case when applying left compression instead of right compression. So we see that the proofs of the appropriate left hand sides become apparent at once on reading the expressions in (2.2), (2.5), (2.8), and (2.9). This is not the case, however, for (2.6) and (2.7) because of the different inequalities satisfied by H_n/x_1 and H_n/x_n. It is for this reason that we have not been able to find suitable expressions for the left hand sides of (1.3) and (1.4).

3. PROOF OF THEOREM 1 FOR $p = 1$

When $p = 1$ the proof of Theorem 1 is very direct (although it is included as a special case of the general proof below). For this reason we present a separate proof for this case. We use the technique which we have referred to above as "compression from the right" and so we put

$$B_n = \mathrm{A}_n(x)\mathrm{G}_n(y) - \mathrm{A}_n(y)\mathrm{G}_n(x),$$

where $\mathrm{A}_n(x)$ denotes the arithmetic mean of x_k with weights w_k etc. We suppose at the outset that $x_1 < x_2 < \cdots < x_n$ and we recall that $y_k = 1 - x_k$.

Keeping all x_k fixed except for x_n and differentiating with respect to x_n we get

$$\frac{dB_n}{dx_n} = w_n\mathrm{G}_n(y) - \mathrm{A}_n(x)\frac{w_n}{y_n}\mathrm{G}_n(y) + w_n\mathrm{G}_n(x) - \mathrm{A}_n(y)\frac{w_n}{x_n}\mathrm{G}_n(x)$$

which simplifies to

$$w_n\left[\frac{\mathrm{G}_n(y)}{y_n}\right][1 - x_n - \mathrm{A}_n(x)] - w_n\left[\frac{\mathrm{G}_n(x)}{x_n}\right][1 - x_n - \mathrm{A}_n(x)] \quad (3.1)$$

(using $\mathrm{A}_n(y) = 1 - \mathrm{A}_n(x)$).

Since $0 < x_n$, $\mathrm{A}_n(x) < \frac{1}{2}$ then $1 - x_n - \mathrm{A}_n(x)$ is positive. And, since $\mathrm{G}_n(x)/\mathrm{G}_n(y) < x_n/y_n$ because $x_k/(1 - x_k)$ increases with k, the expression in (3.1) is positive.

So, as previously, we let $x_n \to x_{n-1}$ from the right (and consequently $y_n \to y_{n-1}$ from the left) and we get

$$B_n > B_{n-1}.$$

We now proceed exactly as in the proof of the right hand side of (1.1), using the notations in (2.4) and get

$$B_n > B_{n-1} > \cdots > B_1.$$

Since $B_1 = 0$ the proof is finished for the case of distinct x_k. Finally, the relaxation of this condition and the conclusion that that inequality is strict, unless all the x_k are equal, follows as before.

4. PROOF OF THEOREM 1 FOR $p \geq 1$

Let us write

$$D_n = \mathrm{A}_n(x)\mathrm{G}_n(y) - \mathrm{A}_n(y)\mathrm{G}_n(x), \tag{4.1}$$

where, in this case, it seems best to parametrize the x_k and y_k as follows (dropping the subscripts for the moment, for brevity):

$$x = \frac{t}{(1+t^p)^{1/p}} \quad \text{and} \quad y = \frac{1}{(1+t^p)^{1/p}} \quad (0 < t < 1). \tag{4.2}$$

Suppose that $t_1 < t_2 < \cdots < t_n$ so that $x_1 < x_2 < \cdots < x_n$ and $y_1 > y_2 > \cdots > y_n$. We keep all the t_k fixed except for t_n and then differentiate with respect to t_n.

If $^{\bullet} \equiv d/dt_n$, (4.1) gives

$$\begin{aligned} \dot{D}_n &= w_n \dot{x}_n \mathrm{G}_n(y) + \mathrm{A}_n(x)\frac{w_n}{y_n}\dot{y}_n \mathrm{G}_n(y) \\ &\qquad - w_n \dot{y}_n \mathrm{G}_n(x) - \mathrm{A}_n(y)\frac{w_n}{x_n}\dot{x}_n \mathrm{G}_n(x) \\ &= w_n \frac{\mathrm{G}_n(y)}{y_n}[\dot{x}_n y_n + \mathrm{A}_n(x)\dot{y}_n] - w_n \frac{\mathrm{G}_n(x)}{x_n}[x_n \dot{y}_n + \mathrm{A}_n(y)\dot{x}_n]. \end{aligned} \tag{4.3}$$

Since $\dot{y}_n$ is negative (see (4.2)) the first bracket here exceeds $\dot{x}_n y_n + x_n \dot{y}_n$ which is strictly positive (see (4.2)) and, since $\mathrm{G}_n(y)/y_n > \mathrm{G}_n(x)/x_n$, the expression in (4.3) strictly exceeds

$$w_n \frac{\mathrm{G}_n(x)}{x_n}\left[\dot{x}_n y_n - x_n \dot{y}_n + \mathrm{A}_n(x)\dot{y}_n - \mathrm{A}_n(y)\dot{x}_n\right]$$

$$= w_n \frac{\mathrm{G}_n(x)}{x_n}\left[\dot{x}_n y_n - x_n \dot{y}_n - \sum_{k=1}^{n} w_k\left(\dot{x}_n y_k - x_k \dot{y}_n\right)\right].$$

It can be seen—again from (4.2)—that the terms in the summation here increase with k and so this last expression exceeds

$$w_n \frac{\mathrm{G}_n(x)}{x_n}\left[\dot{x}_n y_n - x_n \dot{y}_n - \left(\dot{x}_n y_n - x_n \dot{y}_n\right)\right] = 0.$$

Since $\dot{D}_n$ is strictly positive, when we let $t_n \to t_{n-1}$ (from the right) we find that

$$D_n > D_{n-1}.$$

We now continue as before (using the notations of (2.4)) getting

$$D_n > D_{n-1} > \cdots > D_1 = 0$$

thus completing the proof for the case of x_k all distinct. The rest of the proof proceeds as before. And this completes the proof of Theorem 1.

5. FINAL NOTES

(1) In the case $p = 2$ of Theorem 1 the result takes a particularly nice form. With an obvious parametrization it reads

$$\frac{\sum_{k=1}^{n} w_k \sin\theta_k}{\sum_{k=1}^{n} w_k \cos\theta_k} \geq \prod_{k=1}^{n}\left[\tan\theta_k\right]^{w_k} \text{ when } 0 < \theta_k < \frac{\pi}{4}.$$

(2) The method used in this paper bears a superficial resemblance to the λ-method of Mitrinovic and Vasic (see [7]) but it is not the same. In the λ-method (so-called because of the introduction of a parameter λ—to be chosen later) the technique is to apply the method of maxima and minima to a chosen expression which involves the numbers $x_1, x_2, \ldots, x_n$. This expression is treated as a function of x_n only and differentiation is

carried out twice, along with substitution of the critical value of x_n back into the original expression. A similar method is to be found in [3].

This is quite different from the present method in which all that it required is to discover whether (in the case of "right compression") the expression in question increases as x_n increases in $(x_{n-1}, +\infty)$. So differentiation once is all that is required and there is no back substitution. The λ-method, if applied to (2.1) for example, would require the expression in (2.2) to be equated to zero and solved for x_n.

It is interesting to apply both methods to a simple expression, say $A_n - G_n$, to find that two different results are obtained.

(3) Only the means A_n, G_n, and H_n have been considered here. It would be interesting to find similar bounds for differences of the generalized means

$$\mathfrak{M}_n^{[r]} \equiv \left[\sum_1^n w_k x_k^r \right]^{1/r}.$$

If, for simplicity, we write $P_n(r)$ for $\mathfrak{M}_n^{[r]}$ and examine (1.1)–(1.6), one is led to the conjecture that in case $r > s$, the following may be true:

$$\frac{1}{2x_1}\left[\frac{P_n(r)}{x_1}\right]^{1-r} \sum_1^n w_k [x_k - P_n(r)]^2 \geq P_n(r) - P_n(s) \geq \frac{1}{2x_n}\left[\frac{P_n(r)}{x_n}\right]^{1-r} \sum_1^n w_k [x_k - P_n(r)]^2$$

$$\frac{1}{2x_1}\left[\frac{P_n(r)}{x_1}\right]^{1-r} \sum_1^n w_k [x_k - P_n(s)]^2 \geq P_n(r) - P_n(s) \geq \frac{1}{2x_n}\left[\frac{P_n(r)}{x_n}\right]^{1-r} \sum_1^n w_k [x_k - P_n(s)]^2.$$

However, we have not been able to make any progress in this direction.

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